# Lecture 7:
# Dynamic sampling

# Dimension Reduction

# Plan

- Admin:
  - PSet 2 released later today, due next Wed
  - Alex office hours: Tue 2:30-4:30

- Plan:
  - Dynamic streaming graph algorithms
  - **S2**: Dimension Reduction & Sketching

- Scriber?

# Sub-Problem: dynamic sampling

- Stream: general updates to a vector $x \in \{-1,0,1\}^n$

- Goal:
  - Output $i$ with probability $\frac{|x_i|}{\sum_j |x_j|}$

# Dynamic Sampling

- Goal: output $i$ with probability $\frac{|x_i|}{\sum_j |x_j|}$
- Let $D = \{i \text{ s.t. } x_i \neq 0\}$
- Intuition:
  - Suppose $|D| = 10$
    - How can we sample $i$ with $x_i \neq 0$?
    - Each $x_i \neq 0$ is a $1/10$-heavy hitter
    - Use CountSketch ⇒ recover all of them
    - $O(\log n)$ space total
  - Suppose $|D| = 10\sqrt{n}$
    - Downsample: pick a random set $I \subset [n]$ s.t. $\Pr[i \in I] = \frac{1}{\sqrt{n}}$
    - Focus on substream on $i \in I$ only (ignore the rest)
    - What's $|D \cap I|$ ?
      - In expectation = 10
    - Use CountSketch on the downsampled stream $I$…
  - In general: prepare for all levels

# Basic Sketch

- Hash function $g: [n] \to [n]$
- Let $h(i) = \#\text{tail zeros in } g(i)$
  - $\Pr[h(i) = j] = 2^{-j-1}$ for $j = 0..L-1$ and $L = \log_2 n$
- Partition stream into substreams $I_0, I_1, \ldots I_L$
  - Substream $I_j$ focuses on elements with $h(i) = j$
  - $E\left[|D \cap I_j|\right] = |D| \cdot 2^{-j-1}$
- Sketch: for each $j = 0, \ldots L$,
  - Store $CS_j$: CountSketch for $\phi = 0.01$
  - Store $DC_j$: distinct count sketch for approx=1.1
    - $F_2$ would be sufficient here!
  - Both for success probability $1 - 1/n$

# Estimation

- Find a substream $I_j$

  s.t. $DC_j$ output $\in [1,20]$
  - If no such stream, then FAIL
- Recover all $i \in I_j$ with $x_i \neq 0$ (using $CS_j$)
- Pick any of them at random

Algorithm DynSampleBasic:

Initialize:
  hash function $g: [n] \to [n]$
  $h(i) =$ # tail zeros in $g(i)$
  CountSketch sketches $CS_j$, $j \in [L]$
  DistinctCount sketches $DC_j$, $j \in [L]$

Process(int $i$, real $\delta_i$):
  Let $j = h(i)$
  Add $(i, \delta_i)$ to $CS_j$ and $DC_j$

Estimator:
  Let j be s.t. $DC_j \in [1,20]$
  If no such j, FAIL
  $i =$ random heavy hitter from $CS_j$
  Return $i$

# Analysis

- If $|D| < 10$
  - then $|D \cap I_j| \in [1,10]$ for some $j$
- Suppose $D \geq 10$
  - Let $k$ be such that $|D| \in [10 \cdot 2^k, 10 \cdot 2^{k+1}]$
- $E[|D \cap I_k|] = |D| \cdot 2^{-k-1} \in [5,10]$
- $Var[|D \cap I_k|] \leq |D| \cdot 2^{-k-1} \leq 10$
- Chebyshev: $|D \cap I_k|$ deviates from expectation by $4 > \sqrt{1.5 Var}$ with probability at most $\frac{1}{1.5} < 0.7$
  - Ie., probability of FAIL is at most 0.7

Algorithm DynSampleBasic:

Initialize:
  hash function $g: [n] \to [n]$
  $h(i) =$ # tail zeros in $g(i)$
  CountSketch sketches $CS_j$, $j \in [L]$
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Process(int $i$, real $\delta_i$):
  Let $j = h(i)$
  Add $(i, \delta_i)$ to $CS_j$ and $DC_j$

Estimator:
  Let j be s.t. $DC_j \in [1,20]$
  If no such j, FAIL
  $i =$ random heavy hitter from $CS_j$
  Return $i$

# Analysis (cont)

- Let $j$ with $DC_j \in [1,20]$
  - All heavy hitters = $D \cap I_j$
  - $CS_j$ will recover a heavy hitter, i.e., $i \in D \cap I_j$
- By symmetry, once we output some $i$, it is random over $D$
- Randomness?
  - We just used Chebyshev ⇒ pairwise $g$ is OK !

Algorithm DynSampleBasic:

Initialize:
  hash function $g:[n] \to [n]$
  $h(i) =$ # tail zeros in $g(i)$
  CountSketch sketches $CS_j$, $j \in [L]$
  DistinctCount sketches $DC_j$, $j \in [L]$

Process(int $i$, real $\delta_i$):
  Let $j = h(i)$
  Add $(i, \delta_i)$ to $CS_j$ and $DC_j$

Estimator:
  Let j be s.t. $DC_j \in [1,20]$
  If no such j, FAIL
  $i =$ random heavy hitter from $CS_j$
  Return $i$

# Dynamic Sampling: overall

- DynSampleBasic guarantee:
  - FAIL: with probability $\leq 0.7$
  - Otherwise, output a random $i \in D$
    - Modulo a negligible probability of $CS/DC$ failing
- Reduce FAIL probability?
- DynSample-Full:
  - Take $k = O(\log n)$ independent DynSampleBasic
  - Will not FAIL in at least one with probability at least $1 - 0.7^k \geq 1 - 1/n$
  - Space: $O(\log^4 n)$ words for:
    - $k = O(\log n)$ repetitions
    - $O(\log n)$ substreams
    - $O(\log^2 n)$ for each $CS_j, DC_j$

# Back to Dynamic Graphs

- Graph $G$ with edges inserted/deleted
- Define node-edge incidence vectors:
  - For node $v$, we have vector:
    - $x_v \in R^p$ where $p = \binom{n}{2}$
    - For $j > v$: $x_v(v,j) = +1$ if edge $(v,j)$ exists
    - For $j < v$: $x_v(j,v) = -1$ if edge $(j,v)$ exists

| | | $(i,j)$ | | | |
|---|---|---|---|---|---|
| $i$ | | +1 | | | |
| $j$ | | -1 | | | |

- Idea:
  - Use Dynamic-Sample-Full to sample an edge from each vertex $v$
  - Collapse edges
  - How to iterate?
- Property:
  - For a set $Q$ of nodes
  - Consider: $\sum_{v \in Q} x_v$
  - Claim: has non-zero in coordinate $(i,j)$ iff edge $(i,j)$ crosses from $Q$ to outside (i.e., $|Q \cap \{i,j\}| = 1$)
- Sketch enough for: for any set $Q$, can sample an edge from $Q$ !

# Dynamic Connectivity

$x_v \in R^p$ where $p = \binom{n}{2}$
for $j > v$: $x_v(v,j) = +1$ if $\exists(v,j)$
for $j < v$: $x_v(j,v) = -1$ if $\exists(j,v)$

- Sketching algorithm:
  - Dynamic-Sample-Full for each $x_v$
- Check connectivity:
  - Sample an edge from each node $v$
  - Contract all sampled edges
  - $\Rightarrow$ partitioned the graph into a bunch of components $Q_1, \ldots Q_l$ (each is connected)
  - Iterate on the components $Q_1, \ldots Q_l$
- How many iterations?
  - $O(\log n)$ - each time we reduce the number of components by a factor $\geq 2$
- Issue: iterations not independent!
  - Can use a fresh Dynamic-Sampling-Full for each of the $O(\log n)$ iterations

# A little history

- [Ahn-Guha-McGregor'12]: the above streaming algorithm
  - Overall $O(n \cdot \log^4 n)$ space
- [Kapron-King-Mountjoy'13]:
  - Data structure for maintaining graph connectivity under edge inserts/deletes
    - First algorithm with $(\log n)^{O(1)}$ time for update/connectivity !
    - Open since '80s

# Section 2:

# Dimension Reduction & Sketching

# Why?

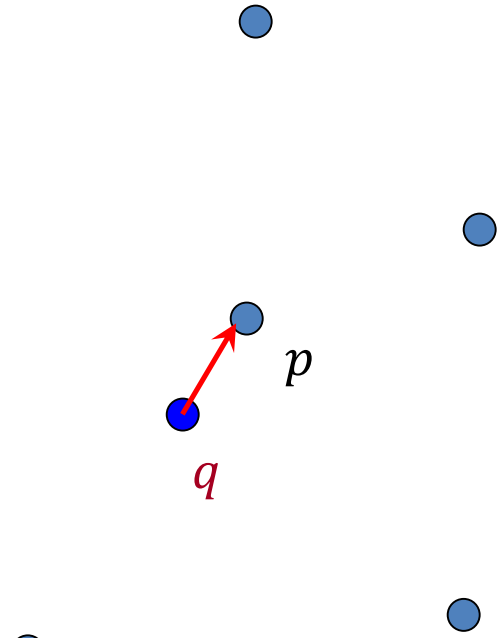
- Application:

  Nearest Neighbor Search

  in high dimensions

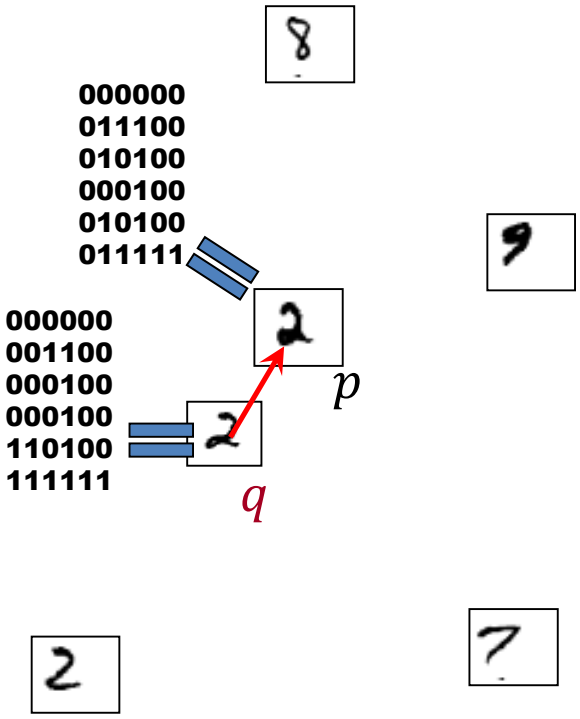
- Preprocess: a set $D$ of points

- Query: given a query point $q$, report a point $p \in D$ with the smallest distance to $q$



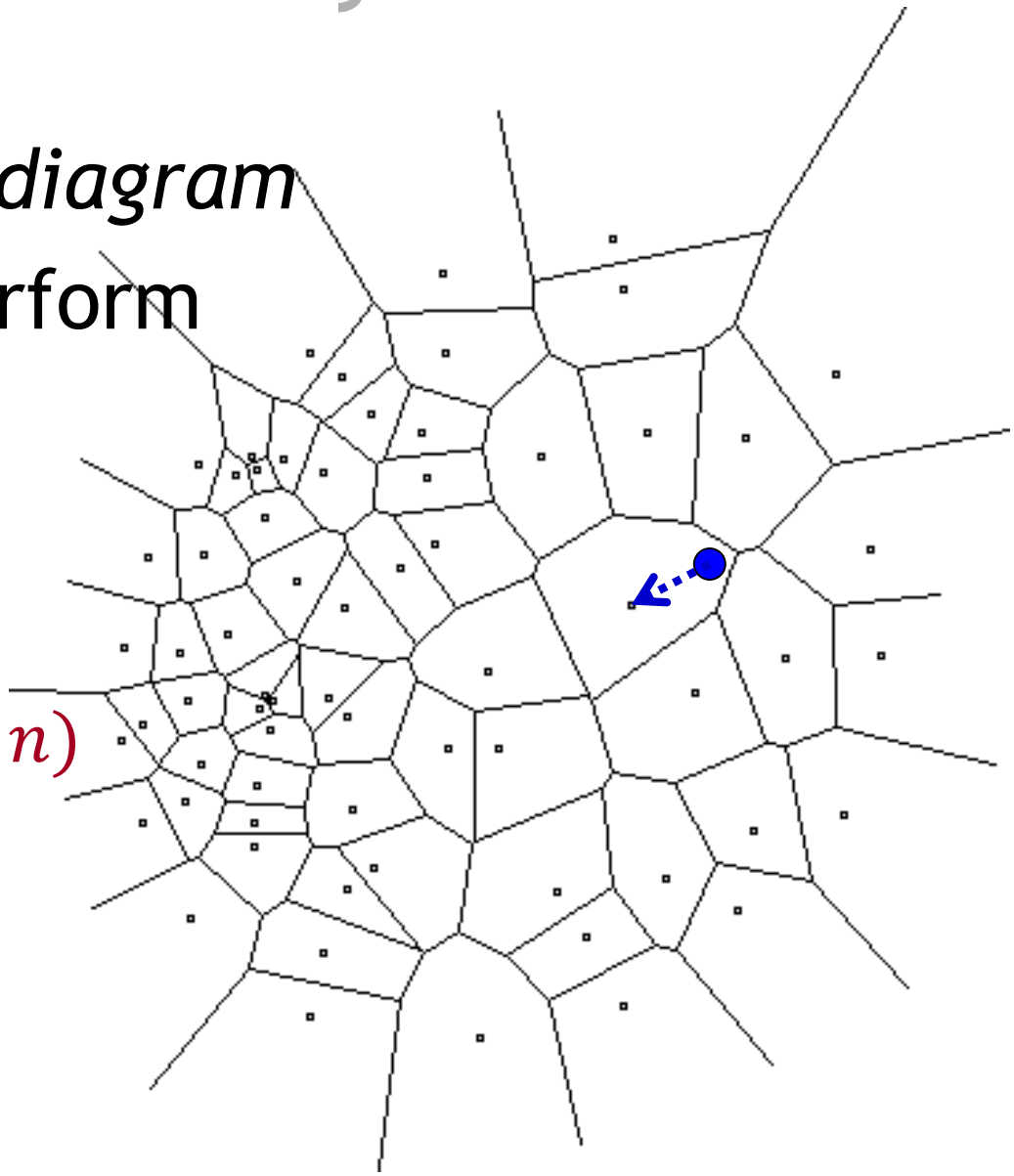
# Motivation

- Generic setup:
  - Points model *objects (e.g. images)*
  - Distance models *(dis)similarity measure*
- Application areas:
  - machine learning: k-NN rule
  - speech/image/video/music recognition, vector quantization, bioinformatics, etc…
- Distance can be:
  - Euclidean, Hamming



# Low-dimensional: easy

- Compute *Voronoi diagram*
- Given query $q$, perform *point location*
- Performance:
  - Space: $O(n)$
  - Query time: $O(\log n)$



# High-dimensional case

- All exact algorithms degrade rapidly with the dimension $d$

| *Algorithm* | *Query time* | *Space* |
|---|---|---|
| Full indexing | $O(\log n \cdot d)$ | $n^{O(d)}$ (Voronoi diagram size) |
| No indexing – linear scan | $O(n \cdot d)$ | $O(n \cdot d)$ |

# Dimension Reduction

- Reduce high dimension?!
  - "flatten" dimension $d$ into dimension $k \ll d$
- Not possible in general: packing bound
- But can if: for a fixed subset of $\Re^d$